The q-Binomial Theorem

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Abstract.
We prove the infinite q-binomial theorem as a consequence of the finite q-binomial theorem.

1. The Finite q-Binomial Theorem

Let x and q be complex numbers, (they can be thought of as real numbers if the reader prefers,) and for the moment, $q \neq 1$. The finite q-binomial theorem is,

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} \frac{(q)_n}{(q)_k(q)_{n-k}} q^{k(k-1)/2} x^k, \tag{1}$$

where n is a positive integer and

$$(q)_k := \begin{cases} (1-q)(1-q^2)\cdots(1-q^k), & \text{if } k = 1,2,3,\ldots, \\ 1, & \text{if } k = 0. \end{cases} \tag{2}$$

Observe that

$$\begin{aligned} \lim_{q\to 1} \frac{(q)_n}{(q)_k(q)_{n-k}} &= \lim_{q\to 1} \frac{(1-q^n)}{(1-q)} \frac{(1-q^{n-1})}{(1-q^2)} \cdots \frac{(1-q^{n+1-k})}{(1-q^k)} \\ &= \frac{n}{1} \cdot \frac{n-1}{2} \cdots \frac{n-k+1}{k} \\ &= \frac{n!}{k!(n-k)!}. \end{aligned}$$

Therefore identity (1) reduces to the well-known binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$$

in the case $q \to 1$.

2. A Proof of the Finite q-Binomial Theorem

In this section, we reproduce the proof of the finite q-binomial theorem given by G. Polya and G.L. Alexanderson in their very interesting paper [4]. Denote the left hand side of (1) by $f(x)$ and observe that

$$(1+x)f(qx) = f(x)(1+q^n x). \tag{3}$$

If we write

$$f(x) = \sum_{k=0}^{n} Q_k x^k,$$

then (3) shows that

$$(1+x)\sum_{k=0}^{n} Q_k q^k x^k = (1+q^n x)\sum_{k=0}^{n} Q_k x^k.$$

Comparing coefficients of x^k, we find that, for $k \geq 1$,

$$Q_k q^k + Q_{k-1} q^{k-1} = Q_k + q^n Q_{k-1},$$

or

$$Q_k = Q_{k-1}\frac{q^{n-k+1}-1}{q^k-1}q^{k-1}.$$

Since $Q_0 = 1$, we conclude that

$$Q_k = \frac{(q)_n}{(q)_k (q)_{n-k}} q^{k(k-1)/2}.$$

3. Two identities of Euler

From now on we will assume that $|q| < 1$. Letting $n \to \infty$ in the finite q-binomial theorem (1) and applying Tannery's theorem [5, §49] to justify letting $n \to \infty$ under the summation sign, we deduce an identity of Euler:

$$\prod_{k=0}^{\infty}(1+q^k x) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{(q)_k}x^k. \tag{4}$$

If we set $x = -1$ in the finite q-binomial theorem (1) and multiply by $(-1)^n/(q)_n$ we find that

$$\sum_{k=0}^{n}\frac{1}{(q)_k (q)_{n-k}} q^{k(k-1)/2}(-1)^{n-k} = 0 \qquad \text{for } n = 1, 2, 3, \ldots. \tag{5}$$

For $n = 0$, it is clear that the expression on the left side of (5) is equal to 1. It is known that if

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

then

$$C(x) = A(x)B(x)$$

where

$$A(x) = \sum_{k=0}^{\infty} a_k x^k, \quad B(x) = \sum_{k=0}^{\infty} b_k x^k \quad \text{and} \quad C(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Consequently, (5) implies that

$$A(x)B(x) = 1$$

where

$$A(x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q)_k} x^k$$

and

$$B(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(q)_k}.$$

This, together with (4), implies another identity of Euler:

$$\prod_{k=0}^{\infty} \frac{1}{(1+q^k x)} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(q)_k}. \tag{6}$$

Equation (4) is valid for all complex numbers x, while equation (6) is valid for $|x| < 1$. In each case $|q| < 1$.

4. The Infinite q-Binomial Theorem

Replace x with $-a$ in the finite q-binomial theorem (1) and write the result in the form

$$\frac{(1-a)(1-qa)\cdots(1-q^{n-1}a)}{(q)_n} = \sum_{k=0}^{n} \frac{q^{k(k-1)/2}(-a)^k}{(q)_k (q)_{n-k}}.$$

By applying the technique in the previous section, we find that

$$\sum_{k=0}^{\infty} \frac{(1-a)(1-qa)\cdots(1-q^{k-1}a)}{(q)_k} x^k = \left(\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q)_k} (-ax)^k \right) \left(\sum_{k=0}^{\infty} \frac{x^k}{(q)_k} \right).$$

By Euler's identities (4) and (6), we deduce

$$\sum_{k=0}^{\infty} \frac{(1-a)(1-qa)\cdots(1-q^{k-1}a)}{(q)_k} x^k = \prod_{k=0}^{\infty} \frac{(1-q^k ax)}{(1-q^k x)}. \tag{7}$$

This is called the infinite q-binomial theorem. It is valid for $|x| < 1$, $|q| < 1$ and any complex number a. If we replace a with q^a and let $q \to 1^-$ in the infinite q-binomial theorem, then we formally obtain the binomial series

$$\sum_{k=0}^{\infty} \frac{(a)(a+1)\cdots(a+k-1)}{k!} x^k = (1-x)^{-a}.$$

A rigorous justification of the limit process is given in [2, pp. 491–492].

The finite q-binomial theorem (1) is a special case of the infinite q-binomial theorem (7). To see this, let n be a positive integer, let $a = q^{-n}$ and replace x with $-q^n x$ in (7). The result is (1). The Euler identities (4) and (6) are also special cases of the infinite q-binomial theorem (7).

In this section we have seen that the infinite q-binomial theorem (7) is in fact a consequence of the finite q-binomial theorem (1).

5. The Jacobi triple product identity

In Section 3 we saw that Euler's identity (4) is a limiting case of the finite q-binomial theorem (1). There is another important limiting case of the finite q-binomial theorem, which we shall now obtain.

In the finite q-binomial theorem (1), replace n with $2n$ and q with q^2, and then let $x = q^{1-2n}z$, to get

$$(1+q^{1-2n}z)\cdots(1+q^{-1}z)(1+qz)\cdots(1+q^{2n-1}z) = \sum_{k=0}^{2n} \frac{(q^2)_{2n}}{(q^2)_k(q^2)_{2n-k}} q^{k^2-2nk} z^k.$$

Multiply both sides by $(q^2)_n q^{n^2} z^{-n}$ and set $k = n + j$ in the sum, to get

$$\prod_{j=1}^{n}(1+q^{2j-1}z)(1+q^{2j-1}z^{-1})(1-q^{2j}) = \sum_{j=-n}^{n} \frac{(q^2)_{2n}(q^2)_n}{(q^2)_{n+j}(q^2)_{n-j}} q^{j^2} z^j.$$

Applying Tannery's theorem [5, §49], we take the limit as $n \to \infty$. We obtain

$$\prod_{j=1}^{\infty}(1+q^{2j-1}z)(1+q^{2j-1}z^{-1})(1-q^{2j}) = \sum_{j=-\infty}^{\infty} q^{j^2} z^j.$$

This result is called the Jacobi triple product identity. It is valid for $|q| < 1$ and any non-zero complex number z. The right hand side should be viewed as a Laurent series in the annulus $0 < |z| < \infty$. There is an essential singularity at $z = 0$. The left hand side shows that there are zeros at $z = \ldots, -q^{-3}, -q^{-1}, -q, -q^3, \ldots$, and these are the only zeros.

The above proof of the Jacobi triple product can be found in [2, p.497]. A completely different proof of the Jacobi triple product identity was given independently by G.E. Andrews [1] and P.K. Menon [3]. Andrews and Menon showed that the Jacobi triple product identity may be proved using the two Euler identities (4) and (6). Since we have shown that the Euler identities are consequences of the finite q-binomial theorem (1), the proof of Andrews and Menon gives another proof of the Jacobi triple product identity which depends only on the finite q-binomial theorem.

6. Summary

In this article we have seen that the two Euler identities (4) and (6), the infinite q-binomial theorem (7) and the Jacobi triple product identity are all consequences of the finite q-binomial theorem (1).

Historical information about the origins of these identities is given in [2, pp. 491, 497 and 501]. Applications of the identities are wide and varied. For example, see [2, Chapters 10 and 11].

References

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